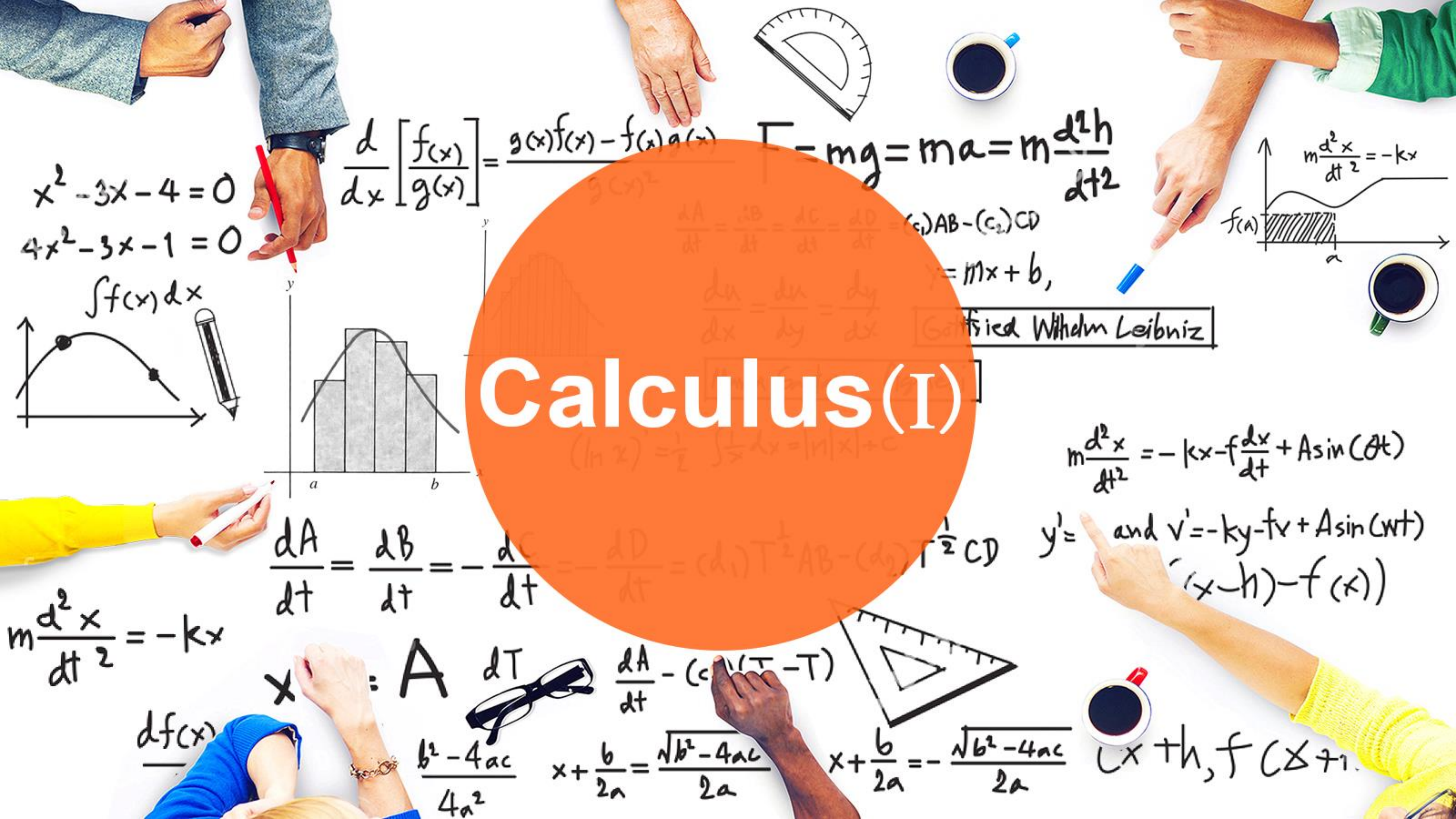

Calculus(Ⅰ)

2.2 The Derivative

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Problem Introduction

Physics: ? The instantaneous velocity of linear motion

Find the instantaneous speed: $v(t_0)$ at time t_0 .

Satisfying the distance function: $s = s(t)$

$t_0 \to t_0 + \Delta t,$

The distance $\Delta s = s(t_0 + \Delta t) - s(t_0),$

Mean velocity $\bar{v}(\Delta t) = \dfrac{\Delta s}{\Delta t}$

Problem Introduction

Case 1: uniform motion

$$v(t_0) = \bar{v}(\Delta t) = \frac{\Delta s}{\Delta t}$$

Case 2： non-uniform motion

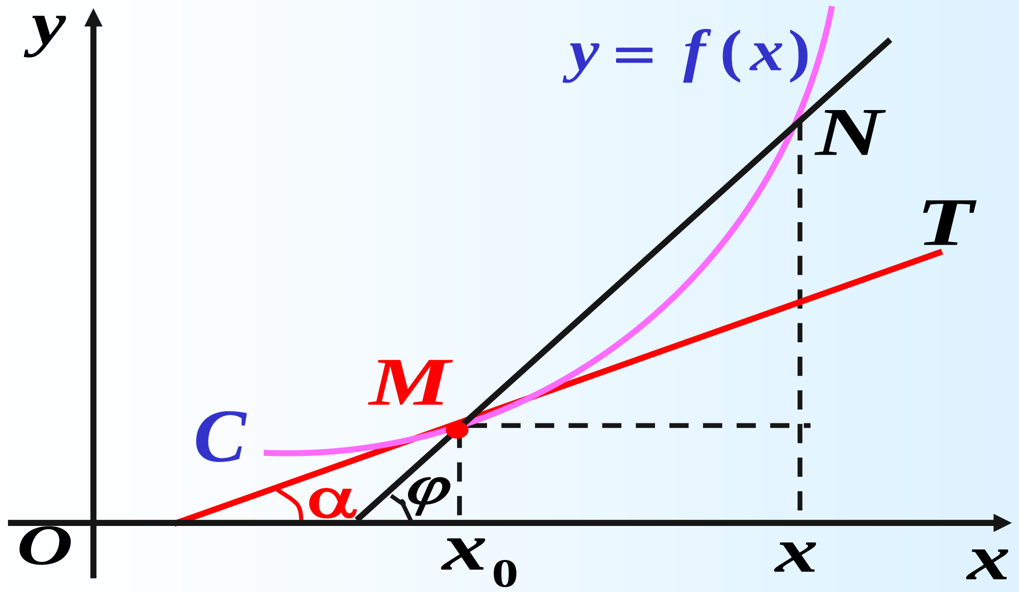
$$v(t_0) = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \lim_{\Delta t \to 0} \frac{s(t_0 + \Delta t) - s(t_0)}{\Delta t}$$

It is called the instantaneous velocity $v(t_0)$ at time t_0

Problem Introduction

Geometry: ? **The tangent line of some curve.**

Tangent line *MT* is the limit position of the segment of *MN* if *N* tends to *M*.



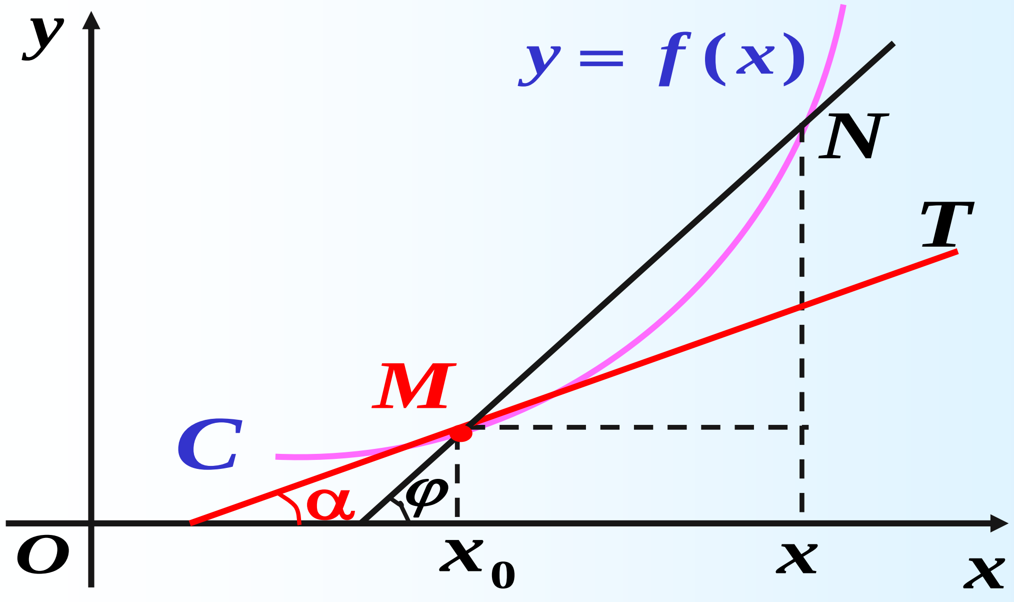
The limit position

$$|MN| \to 0, \quad \angle NMT \to 0.$$

Let $y = f(x)$. Find the slope of tangent line at point $M_0(x_0, y_0)$.

Problem Introduction

$\because$ Let $M(x_0, y_0)$, $N(x, y)$.



The slope of MN:

$$\tan\varphi = \frac{y - y_0}{x - x_0} = \frac{f(x) - f(x_0)}{x - x_0}$$

$N \xrightarrow{\text{along curve C}} M$,

$x \to x_0$,

$\therefore$ The slope of tangent MT

$$k = \tan\alpha$$

$$= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}.$$

Definition of the Derivative

Definition: Derivative

The derivative of a function f is another function f'(read "f prime") whose value at any number x is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

If the limit does exist,we say f is differentiable at x.

Definition of the Derivative

Remark 1: For any point x, written in some forms:

(1) $$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

(2) $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

(3) $$f'(x) = \lim_{-h \to 0} \frac{f(x-h) - f(x)}{-h}$$

Definition of the Derivative

Remark 2: For some fixed point x_0 , written in some forms:

(4) $$f'(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

(5) $$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

(6) $$f'(x_0) = \lim_{-h \to 0} \frac{f(x_0 - h) - f(x_0)}{-h}$$

$$f'(x_0) = f'(x)\Big|_{x=x_0}$$

If $x_0 = 0$, $f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x}$.

Theorem A: Differentiability Implies Continuity

If $f'(c)$ exists, then f is continuous at c.

Need to show that $\lim_{x \to c} f(x) = f(c)$.

$$\because \frac{f(x) - f(c)}{x - c} = \frac{f(x) - f(c)}{x - c} \text{ has mathematical meaning}$$

$$\therefore f(x) = f(c) + \frac{f(x) - f(c)}{x - c} \cdot (x - c)$$

Therefore,

$$\lim_{x \to c} f(x) = \lim_{x \to c}\left[f(c) + \frac{f(x) - f(c)}{x - c} \cdot (x - c) \right]$$

$$= \lim_{x \to c} f(c) + \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \cdot \lim_{x \to c}(x - c)$$

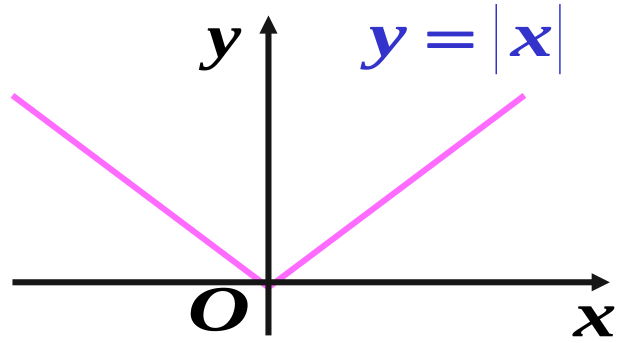
$$= f(c) + f'(c) \cdot 0 = f(c).$$

Note

The inverse theorem is not ture.

Eg, Determine $f(x)=|x|$ is continuous or not at $x=0$.

$x=0$ not differentiable.



Def:

(left derivative)

$$f'_{-}(x_0) = \lim_{x \to x_0 - 0} \frac{f(x) - f(x_0)}{x - x_0}$$

(right derivative)

$$f'(x_0 - 0) = \lim_{\Delta x \to -0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x};$$

$$f'_{+}(x_0) = \lim_{x \to x_0 + 0} \frac{f(x) - f(x_0)}{x - x_0}$$

$$f'(x_0 + 0) = \lim_{\Delta x \to +0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}.$$

Geometrically $y = f(x)$ at $(x_0, f(x_0))$

The slope of left tangent line = The slope of right tangent line

$f'(x_0)$ exists $\Leftrightarrow$ $f'(x_0 + 0)$ and $f'(x_0 - 0)$ both exsit and equal.

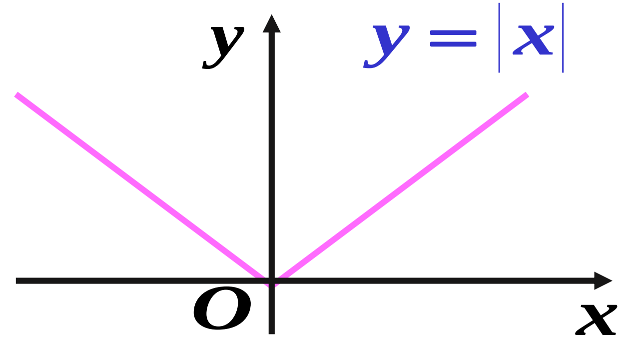
Def:

Determine $f(x)=|x|$ is differentiable or not at $x=0$.

$$\because \frac{f(0+h)-f(0)}{h}=\frac{|h|}{h},$$

$$\lim_{h\to 0^+}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0^+}\frac{h}{h}=1,$$

$$\lim_{h\to 0^-}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0^-}\frac{-h}{h}=-1.$$



Namely $f'_+(0)\neq f'_-(0)$,

$\therefore y=f(x)$ NOT Differentiable at $x=0$.

Example 1

If $f(x)$ is derivative at $x = a$, find $\lim_{x \to 0} \frac{f(a+3x) - f(a)}{5x}$.

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

$$\lim_{x \to 0} \frac{f(a+3x) - f(a)}{5x} = \lim_{x \to 0} \frac{f(a+3x) - f(a)}{\frac{5}{3} \cdot 3x}$$

$$= \frac{3}{5} \lim_{x \to 0} \frac{f(a+3x) - f(a)}{3x}$$

$$= \frac{3}{5} f'(a).$$

Example 2

If $f = \dfrac{1}{x}$, find $f'(x)$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h}$$

$$= -\frac{1}{x^2}.$$

$$\left(\frac{1}{x}\right)' = -\frac{1}{x^2}.$$

Example 3

Find the derivative of function $f(x) = C$ (C is a constant)

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{C - C}{h}$$

$$= 0.$$

$$(C)' = 0$$

Example 4

Let function $f(x)=\sin x$, find $(\sin x)'$ and $(\sin x)'\big|_{x=\frac{\pi}{4}}$

$$(\sin x)'=\lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}$$

$$=\lim_{h\to 0}\cos\left(x+\frac{h}{2}\right)\cdot\frac{\sin\frac{h}{2}}{\frac{h}{2}}$$

$$=\cos x.$$

$$\therefore (\sin x)'\big|_{x=\frac{\pi}{4}}$$

$$=\cos x\big|_{x=\frac{\pi}{4}}=\frac{\sqrt{2}}{2}.$$

$(\sin x)'=\cos x$

Similarly $(\cos x)'=-\sin x$

Summary of the Derivative

$$(1)\ f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$(2)\ f'(x)=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}$$

$$(3)\ f'(x_0)=\lim_{\Delta x\to 0}\frac{f(x)-f(x_0)}{x-x_0}$$

$$(4)\ f'(1)=\lim_{h\to 0}\frac{f(1+h)-f(1)}{h}$$

$$=\lim_{p\to 0}\frac{f(1+p)-f(1)}{p}$$

$$=\lim_{s\to 0}\frac{f(1+s)-f(1)}{s}$$

Summary of the Derivative

Step 1

Find increment: $\Delta y = f(x+\Delta x) - f(x)$

Step 2

Find ratio: $\dfrac{\Delta y}{\Delta x} = \dfrac{f(x+\Delta x) - f(x)}{\Delta x}$

Step 3

Find limit: $y' = \lim\limits_{\Delta x \to 0} \dfrac{\Delta y}{\Delta x}$

Questions and Answers

If $y=\log_a x\ (a>0, a\neq 1)$, find y'.

$$y'=\lim_{h\to 0}\frac{\log_a(x+h)-\log_a x}{h}$$

$$=\lim_{h\to 0}\frac{\log_a(1+\frac{h}{x})}{\frac{h}{x}}\cdot\frac{1}{x}$$

$$=\frac{1}{x}\lim_{h\to 0}\log_a(1+\frac{h}{x})^{\frac{x}{h}}=\frac{1}{x}\log_a e.$$

$$(\log_a x)'=\frac{1}{x}\log_a e$$

$$(\ln x)'=\frac{1}{x}$$

Questions and Answers

If $f = 13x - 6$, find $f'(4)$.

$$f'(4) = \lim_{h \to 0} \frac{f(4+h) - f(4)}{h}$$

$$= \lim_{h \to 0} \frac{13h}{h}$$

$$= 13.$$

Questions and Answers

If $f=\sqrt{x}, x>0$, find $f'(x)$.

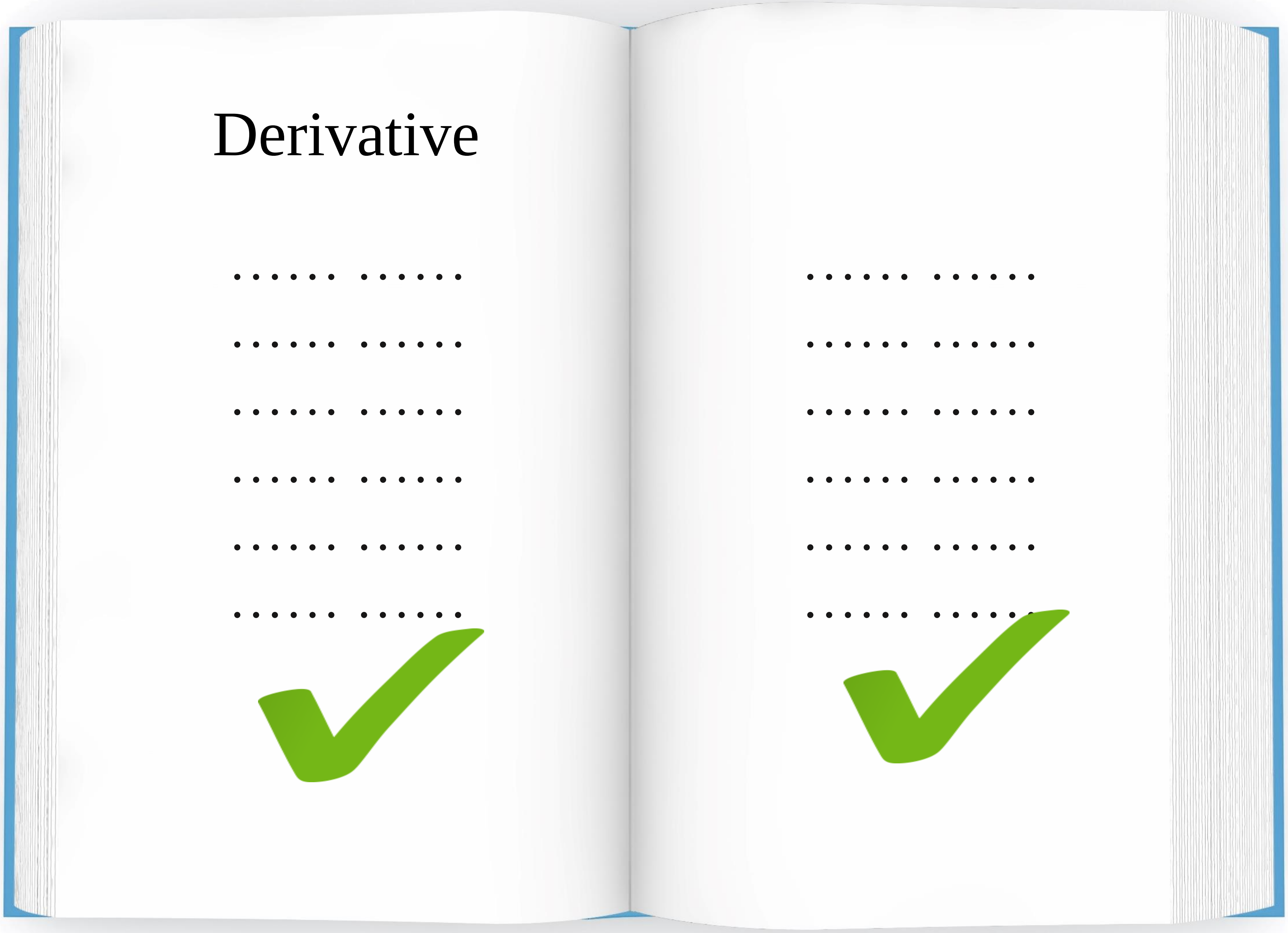
$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\left[\frac{\sqrt{x+h}-\sqrt{x}}{h}\cdot\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}\right]$$

$$=\lim_{h\to \mathbf{0}}\frac{\mathbf{1}}{\sqrt{x+h}+\sqrt{x}}$$

$$=\frac{1}{2\sqrt{x}}.$$

$$(\sqrt{x})'=\frac{1}{2\sqrt{x}}.$$

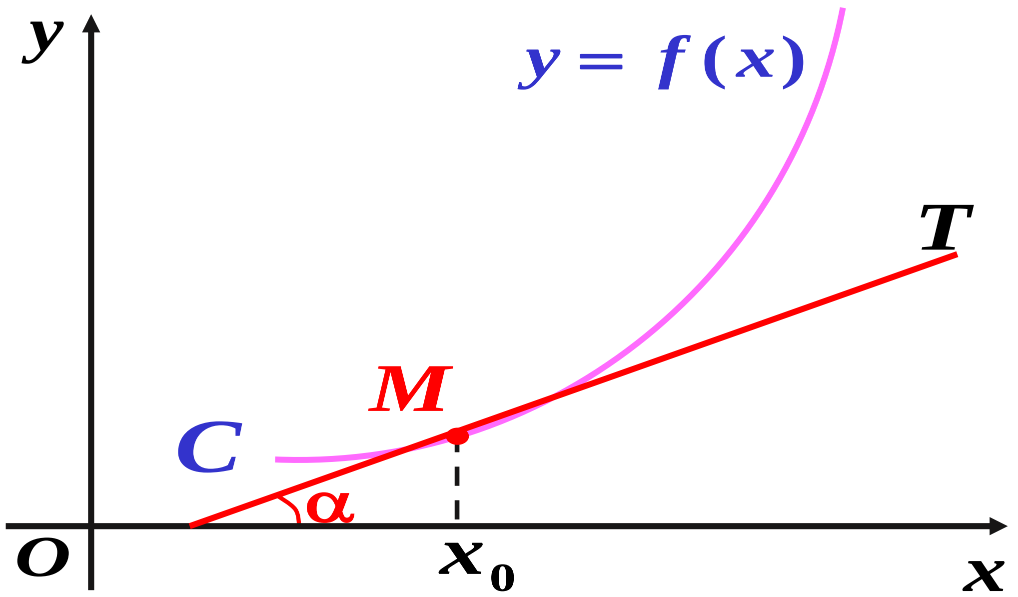

Derivative

Geometrical meaning of derivative

1. Geometrical meaning

$f'(x_0)$ expresses the slope of tangent line.

$$f'(x_0) = \tan\alpha$$



Specially,

(1) If $f'(x_0) = 0,$ tangent line // x-axix

(2) If $f'(x_0) = \infty,$ tangent line $\perp$ x-axix.

Geometrical meaning of derivative

The tangent equation of $y = f(x)$ at $(x_0, f(x_0))$:

$$y - y_0 = f'(x_0)(x - x_0).$$

The normal euqation of $y = f(x)$ at $(x_0, f(x_0))$:

$$y - y_0 = -\frac{1}{f'(x_0)}(x - x_0) \quad \left(f'(x_0) \neq 0 \right).$$

Example

$$y - y_0 = f'(x_0)(x - x_0)$$

Find the tangent equation and normal equation of

$$y = \frac{1}{x} \quad \text{at} \quad (\frac{1}{2}, 2).$$

By geometrical meaning,

$$y - y_0 = -\frac{1}{f'(x_0)}(x - x_0)$$

$$\boldsymbol{k = y'\Big|_{x=\frac{1}{2}} = (\frac{1}{x})'\Big|_{x=\frac{1}{2}} = -\frac{1}{x^2}\Big|_{x=\frac{1}{2}} = -4.}$$

Tangent equation is: $y - 2 = -4(x - \frac{1}{2})$, namely $4x + y - 4 = 0.$

Normal equation is: $y - 2 = \frac{1}{4}(x - \frac{1}{2})$, namely $2x - 8y + 15 = 0.$

Example

Discuss the continuity and differentible at $x = 0$

for the function $f(x) = \begin{cases} x \sin \dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$.

$\because \sin \dfrac{1}{x}$ bounded function, $\therefore \lim\limits_{x \to 0} x \sin \dfrac{1}{x} = 0$

$\because f(0) = \lim\limits_{x \to 0} f(x) = 0$ $\therefore f(x)$ is continuous at $x = 0$.

at $x = 0$, $\dfrac{\Delta y}{\Delta x} = \dfrac{(0 + \Delta x)\sin \dfrac{1}{0 + \Delta x} - 0}{\Delta x} = \sin \dfrac{1}{\Delta x}$,

when $\Delta x \to 0$, $\dfrac{\Delta y}{\Delta x}$ is changeable between -1 and 1.

$\therefore f(x)$ is NOT differentiable at $x = 0$.

Question

If $f(x)=\begin{cases} x^2, & \text{when } x \le x_0, \\ ax+b, & \text{when } x > x_0. \end{cases}$ $f(x)$ is derivative at x_0, how about a,b ?

f(x) is continuous at x_0, by

$$\lim_{x\to x_0^-} f(x)=x_0{}^2, \quad \lim_{x\to x_0^+} f(x)= ax_0+b, \quad f(x_0)=x_0^2.$$

So, $ax_0+b = x_0^2$. And

$$f'_-(x_0)=\lim_{x\to x_0-}\frac{f(x)-f(x_0)}{x-x_0}=\lim_{x\to x_0-}\frac{x^2-x_0{}^2}{x-x_0}= 2x_0$$

Question

If $f(x)=\begin{cases} x^2, & \text{when } x \le x_0, \\ ax+b, & \text{when } x > x_0. \end{cases}$ $f(x)$ is derivative at x_0, how about a,b ?

$$f'_+(x_0)=\lim_{x\to x_0+}\frac{f(x)-f(x_0)}{x-x_0}=\lim_{x\to x_0+}\frac{(ax+b)-x_0^2}{x-x_0}$$

$$=\lim_{x\to x_0+}\frac{(ax+b)-(ax_0+b)}{x-x_0}$$

$$ax_0+b=x_0^2$$

$$=\lim_{x\to x_0+}\frac{ax-ax_0}{x-x_0}=a=2x_0=f'_-(x_0)$$

Then, when $a=2x_0$, $b=-x_0^2$, $f(x)$ is derivative at x_0.